

Exploring an Infinite Space with Finite Memory Scouts

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Abstract

Consider a small number of *scouts* exploring the infinite d -dimensional grid with the aim of hitting a hidden target point. Each scout is controlled by a probabilistic finite automaton that determines its movement (to a neighboring grid point) based on its current state. The scouts, that operate under a fully synchronous schedule, communicate with each other (in a way that affects their respective states) when they share the same grid point and operate independently otherwise. Our main research question is: How many scouts are required to guarantee that the target admits a *finite mean hitting time*? Recently, it was shown that $d + 1$ is an upper bound on the answer to this question for any dimension $d \geq 1$ and the main contribution of this paper comes in the form of proving that this bound is tight for $d \in \{1, 2\}$.

1 Introduction

Model and Contribution. Consider c scouts u^1, \dots, u^c that explore the d -dimensional infinite grid \mathbb{Z}^d . Each scout is controlled by a probabilistic finite automaton with (finite) *state set* \mathcal{S} so that at any given time $n \in \mathbb{Z}_{\geq 0}$, the global system *configuration* is characterized by the scouts' location vector $\mathbf{X}_n \in \mathbb{Z}^{d \times c}$ and state vector $\mathbf{Q}_n \in \mathcal{S}^c$.¹

The scouts *sense* their respective grid points through the following mechanism: At time $n \in \mathbb{Z}_{\geq 0}$, each scout u^i observes its local (binary) *environment* vector $E_n^i \in \{0, 1\}^{\mathcal{S}}$ defined so that $E_n^i(q) = 1$ if and only if there exists some $j \neq i$ such that $X_n^j = X_n^i$ and $Q_n^j = q$. In other words, scout u^i can sense the current state $q \in \mathcal{S}$ of another scout u^j if they reside in the same grid point. Notice that the environment E_n^i is fully determined by the configuration $(\mathbf{X}_n, \mathbf{Q}_n)$.

The scouts' actions are controlled by a (probability) *transition function*²

$$\Pi : (\mathcal{S} \times \{0, 1\}^{\mathcal{S}}) \times (\mathcal{S} \times \{-1, 0, +1\}^d) \rightarrow [0, 1]$$

defined so that $\Pi((q, e), (q', \xi))$ is the probability that at time $n + 1$, scout u^i resides in state $Q_{n+1}^i = q'$ in grid point $X_{n+1}^i = x + \xi$ given that $Q_n^i = q$, $E_n^i = e$, and $X_n^i = x$. The application of the transition function Π for scout u^i at time n is independent of its application for scout u^j at time n' for any $j \neq i$ or $n \neq n'$.³

To complete the model specification, each scout u^i is associated with an *initial state* $q_0^i \in \mathcal{S}$ so that $Q_0^i = q_0^i$. Furthermore, it is assumed that at time 0, all scouts share the same *initial location* $X_0^i = x_0 \in \mathbb{Z}^d$; this is typically taken to be the origin, i.e., $x_0 = 0$, however later on in the technical sections, we also consider scouts with arbitrary initial locations. We refer to the 5-tuple $\langle c, \mathcal{S}, x_0, \mathbf{q}_0, \Pi \rangle$ as a *scout protocol*.

The process $(\mathbf{X}_n, \mathbf{Q}_n)_{n \geq 0}$ will be referred to as a *scout process* (or a *c-scout process*). Given a scout protocol $\mathcal{P} = \langle c, \mathcal{S}, x_0, \mathbf{q}_0, \Pi \rangle$, it is completely defined via

$$(\mathbf{X}_0, \mathbf{Q}_0) = (x_0^c, \mathbf{q}_0)$$

and

$$\mathbb{P}(\mathbf{X}_{n+1} - \mathbf{X}_n = \Xi, \mathbf{Q}_{n+1} = \mathbf{q} \mid (\mathbf{X}_m, \mathbf{Q}_m)_{m=0}^n) = \prod_{i=1}^c \Pi((Q_n^i, E_n^i), (q^i, \Xi^i)) \text{ a.s.}^4$$

¹Throughout, we use the boldfaced notation \mathbf{v} to denote the vector $\mathbf{v} = (v^1, \dots, v^c)$.

²A function $\Pi : \mathcal{A} \times \mathcal{B} \mapsto [0, 1]$, where \mathcal{A} and \mathcal{B} are countable non-empty sets, is called a (probability) transition function or transition kernel if $\sum_{b \in \mathcal{B}} \Pi(a, b) = 1$ for all $a \in \mathcal{A}$.

³The choice of introducing a single transition function Π for all scouts, rather than a separated transition function Π^i for each scout u^i , does not result in a loss of generality, as the space set \mathcal{S} can be partitioned into pairwise disjoint subsets $\mathcal{S} = \mathcal{S}^1 \cup \dots \cup \mathcal{S}^c$, ensuring that $Q_n^i \in \mathcal{S}^i$ for every $1 \leq i \leq c$ and $n \in \mathbb{Z}_{\geq 0}$.

⁴Notice that both sides of the last equation are random variables. It should be interpreted as stating that for every $n, \mathbf{x}_0, \dots, \mathbf{x}_n \in \mathbb{Z}^{d \times c}$, and $\mathbf{q}_0, \dots, \mathbf{q}_n \in \mathcal{S}^c$, if the event $\bigwedge_{m=0}^n (\mathbf{X}_m, \mathbf{Q}_m) = (\mathbf{x}_m, \mathbf{q}_m)$ occurs with positive probability, then for every $\Xi \in \mathbb{Z}^{d \times c}$, and $\mathbf{q} \in \mathcal{S}^c$, the probability, conditioned on that event, that $\mathbf{X}_{n+1} = \mathbf{x}_n + \Xi$ and $\mathbf{Q}_{n+1} = \mathbf{q}$ equals to $\prod_{i=1}^c \Pi(q_n^i, e_n^i, q^i, \Xi^i)$, where e_n^i is the environment of scout u^i with respect to the configuration $(\mathbf{x}_n, \mathbf{q}_n)$.

for all $n \geq 0$, $\Xi \in \mathbb{Z}^{d \times c}$ and $\mathbf{q} \in \mathcal{S}^c$, where Ξ^i denotes the i -th (d -dimensional) component of Ξ .

The *hitting time* of grid point $x \in \mathbb{Z}^d$ under \mathcal{P} is defined as $\inf\{n \geq 0 : \exists i \text{ s.t. } X_n^i = x\}$. The scout protocol \mathcal{P} is said to be *effective* if every grid point $x \in \mathbb{Z}^d$ admits a finite mean hitting time. It turns out that $c = d + 1$ scouts are sufficient for the design of an effective scout protocol on \mathbb{Z}^d for every dimension $d \geq 1$ (see [18, Section 3.2] and the related literature discussion of the present paper). In this paper, we prove that $c = d + 1$ is also a necessary condition for $d \in \{1, 2\}$ and conjecture that this holds for $d \geq 3$ as well.

Theorem. *For $d \in \{1, 2\}$, any effective scout protocol on \mathbb{Z}^d requires at least $d + 1$ scouts.*

Conjecture. *For $d \geq 3$, any effective scout protocol on \mathbb{Z}^d requires at least $d + 1$ scouts.*

Related Work. Graph exploration appears in many forms and is widely studied in the CS/Mathematics literature. In the general graph exploration setting, an *agent* (or a group thereof) is placed in some node of a given graph and the goal is to visit every node (or every edge) by traversing the graph along its edges.⁵ There are plenty of variants and modifications of the graph exploration problem, where one natural classification criterion is to distinguish between *directed* graph exploration [1, 14], where the edge traversals are uni-directional, and *undirected* graph exploration [6, 15, 17], where the edges can be traversed in both directions. The present paper falls into the latter setting.

The graph exploration domain can be further divided into the case where the nodes are labeled with unique identifiers that the agents can recognize [17, 29], and the case where the nodes are anonymous [8, 11, 30]. The exact conditions for a successful exploration also vary between different graph exploration works, where in some papers the agents are required to halt their exploration process [15] and sometimes also return to their starting position(s) [5]. The setting considered in the present paper requires neither halting nor returning to the initial grid point and the nodes are anonymous (in fact, since our scouts are controlled by finite automata that cannot “read” unbounded labels, unique node identifiers would have been meaningless).

A standard measurement for the efficiency of a graph exploration protocol is its time complexity [29]. When the agents are assumed to operate under a synchronous schedule, this measurement typically counts the number of (synchronous) rounds until the exploration process is completed in the worst case. This can be generalized to asynchronous schedules by defining a time unit as the longest delay of any atomic action [8, 12]. A good example for a synchronous search problem concerning the minimization of the worst-case search time is the widely studied *cow path problem*, introduced by Baeza-Yates et al. [7]. This problem involves a near-sighted cow standing at a crossroads with w paths leading to an unknown territory. By traveling at unit speed, the cow wishes to discover a patch or clover that is at distance d from the origin in as small time as possible.

⁵In some research communities, it is common to refer to the exploring agents as *robots*, *ants*, or *particles*. Since the agents considered in this paper differ slightly from those considered in the existing literature, we chose the distinctive term *scouts*.

Baeza-Yates et al. proposed a deterministic algorithm called *linear spiral search* and showed that in the $w = 2$ case, this algorithm will find the goal in time at most $9d$. The classic cow path problem was extended to the multiple cows setting by López-Ortiz and Sweet [28].

Another common measure for the efficiency of a graph exploration protocol is the size of the memory that the agents can maintain [15, 24]. In the present paper, the execution is assumed to be synchronous and the scouts are controlled by (probabilistic) finite automata whose memory size is constant, independent of the distance to the target.

Graph exploration with agents controlled by finite automata has been studied, e.g., by Fraigniaud et al. [25] who showed that a single deterministic agent needs $\Theta(D \log \Delta)$ bits of memory to explore a graph, where D stands for the diameter and Δ stands for the maximum degree of the input graph. Considerable literature on graph exploration by a finite automaton agent focuses on a special class of graphs called *labyrinths* [9, 11, 16]. A labyrinth is a graph that can be embedded on the infinite 2-dimensional grid and has a set of obstructed cells that cannot be entered by the agent. The labyrinth is said to be *finite* if the set of obstructed cells is finite and the term *exploration* in the context of finite labyrinths refers to the agent getting arbitrarily far from its (arbitrary) starting points. It is known that any finite labyrinth can be explored by a deterministic finite automaton agent using 4 *pebbles*, where a pebble corresponds to a movable marker, and some finite labyrinths cannot be explored with one pebble [10]. While the infinite 2-dimensional grid studied in the present paper is a special (somewhat degenerate) case of a finite labyrinth and our scouts are also controlled by finite automata, the goal of the scout protocols considered here is to hit every grid point and we allow randomization.

An extensively studied special case of a single agent controlled by a probabilistic finite automaton is that of a (simple unbiased) *random walk*. Aleliunas et al. [2] proved that for every finite (undirected) graph, the random walk’s *cover time*, i.e., the time taken to visit every node at least once, admits a polynomial mean. Alon et al. [4] studied the extension of this classic model to multiple agents. Among other results, they proved that in some graphs, the speed-up in the mean cover time (as a function of the number of agents) is exponential, whereas in other graphs, it is only logarithmic. Cooper et al. [13] investigated the case of c independent random walks exploring a random r -regular graph, and analyzed the asymptotic behavior of the graph cover time and the number of steps before any of the walks meet.

Random walks receives a lot of attention also in the context of infinite graphs. A classic result in this regard is that a random walk on \mathbb{Z}^d is *recurrent* — namely, it reaches every point with probability one — if and only if $d \leq 2$; nevertheless, even in this case, it is only *null-recurrent*: the mean hitting time of some (essentially all) points is infinite. This gives rise to another research question asking whether the exploration process induced by $c > 1$ (non-interacting) random walks on \mathbb{Z}^d is *positive-recurrent*, that is, the mean hitting time of all points is finite. It is relatively well known (can be derived, e.g., from Proposition 4.2.4 in [27]) that for $d = 1$, the minimum value of c that yields a positive-recurrent exploration process is $c = 3$, whereas for $d = 2$, the

exploration process induced by c random walks remains null-recurrent for any finite c . Cast in this terminology, the research question raised in the present paper is how many agents are required to turn the exploration process on \mathbb{Z}^d (dealing with $d \in \{1, 2\}$) into a positive-recurrent one, given that the agents are augmented with a finite memory logic and local interaction skills.

The power of team exploration and the effect of the agents ability to communicate with each other has been studied in various settings. Fraigniaud et al. [23] investigated the exploration of a tree by c locally interacting mobile agents and constructed a distributed exploration algorithm whose running time is $O(c/\log(c))$ times larger than the optimal exploration time with full knowledge of the tree. They also showed that for some trees, in the absence of communication, every algorithm is $\Omega(c)$ times slower than the optimal. Another good example for the advantages in the agents communication can be found in the work of Bender et al. [8] showing that two cooperating robots that can recognize when they are at the same node and can communicate freely at all times can learn any strongly-connected directed graph with n indistinguishable nodes in expected time polynomial in n .

Exploration processes by agents with limited memory occur also in nature. A prominent example for such a process is *ant foraging* that was modeled recently by Feinerman et al. [21, 20] as an abstract distributed computing task involving a team of n *ants* that explore \mathbb{Z}^2 in search for an adversarially hidden treasure. A variant of this model, where the ants are controlled by finite automata with local interaction capabilities was studied in [19, 26].

While the focus of [19, 26], as well as that of [21, 20], is on the speed-up as n grows asymptotically, Emek et al. [18] investigated the smallest number of ants that can guarantee that the treasure is found in finite time. They studied various different settings (e.g., deterministic ants, ants controlled by pushdown automata, asynchronous schedules), but when restricted to ants controlled by probabilistic finite automata operating under a synchronous schedule, their model is very similar to the one studied in the present paper. Cast in our terminology, they established the existence of an effective scout protocol with 3 scouts on \mathbb{Z}^2 and their proof can be easily extended to design an effective scout protocol with $d+1$ scouts on \mathbb{Z}^d for any fixed $d \geq 1$ (see [18, Theorem 3]). They also claimed that there does not exist any effective scout protocol with 1 scout on \mathbb{Z}^2 (see [18, Theorem 7]), but the proof of this claim admits a significant gap;⁶ the impossibility result established in the present paper regarding the 1-dimensional grid clearly subsumes that claim. For a survey on generalized graph exploration problems, see, e.g., [22].

Paper Organization. The primary technical contribution in this paper is the proof that there does not exist an effective scout protocol on \mathbb{Z}^2 with 2 scouts. This proof is presented in Section 2 with some necessary tools developed in Section 3. It turns out that the line of arguments leading

⁶Specifically, the authors of [18], who has a non-empty intersection with the authors of the present paper, defined in Section 5.1 of that paper a certain subset \mathbb{Z}_s of \mathbb{Z} and claimed that every point in \mathbb{Z}_s has a finite mean hitting time. This claim is not substantiated in [18] and we now understand that its proof is far more demanding than the general intuition we had in mind when [18] was written.

to this proof also includes, in passing, a proof for the secondary result of this paper: there does not exist an effective scout protocol on \mathbb{Z}^1 with 1 scout. For clarity, in Section 4, we refine the proof of this secondary \mathbb{Z}^1 result from that of the primary \mathbb{Z}^2 case.

2 Two Scouts on \mathbb{Z}^2

Our main goal in this section is to establish the following theorem.

Theorem 2.1. *Let $(\mathbf{X}_n, \mathbf{Q}_n)_{n \geq 0}$ be a scout process on \mathbb{Z}^2 with protocol $\mathcal{P} = \langle 2, \mathcal{S}, 0, \mathbf{q}_0, \Pi \rangle$. Then there exists some grid point $x \in \mathbb{Z}^2$ of which the expected hitting time is infinite, namely,*

$$\mathbb{E} \left(\inf \{ n \geq 0 : X_n^1 = x \text{ or } X_n^2 = x \} \right) = \infty. \quad (1)$$

The proof of Theorem 2.1 is presented in a top-down fashion. Its main ideas and arguments are introduced in Section 2.1. The argumentation relies on three non-trivial propositions which are proved in Sections 2.3, 2.4, and 2.5. The proofs of these propositions rely on a generalization of the scout process notion which is presented in Section 2.2. They also make use of some random walk hitting time estimates which are relegated to section 3.

2.1 Top Level Proof

For the remaining of this section, let us suppose towards a contradiction that (1) does not hold. The first step is to show that if this is the case, then the two scouts must meet infinitely often with probability one and moreover, the distribution of the time between successive meetings must have a stretched-exponentially decaying upper tail. Formally, let $N_0 := 0$ and set

$$N_k := \inf \{ n > N_{k-1} : X_n^1 = X_n^2 \}, \quad k = 1, 2, \dots$$

Proposition 2.2. *Let $(\mathbf{X}_n, \mathbf{Q}_n)_{n \geq 0}$ be a scout process on \mathbb{Z}^2 with two scouts, state space \mathcal{S} , transition function Π , initial position $\mathbf{x}_0 = 0$, and initial state $\mathbf{q}_0 \in \mathcal{S}^2$ and suppose that (1) does not hold. Then, with probability one, all N_k are finite. Moreover, there exists $\delta > 0$ such that for all $k = 0, 1, \dots$ and $u \geq 0$, almost surely*

$$\mathbb{P}(N_{k+1} - N_k > u \mid \mathbf{Q}_{N_k}) \leq \frac{1}{\delta} e^{-\delta \sqrt{u}}. \quad (2)$$

In view of Proposition 2.2, we now define for $k = 0, 1, \dots$, the random variables

$$\begin{aligned} \mathbf{A}_k &:= \mathbf{Q}_{N_k}, \\ Y_k &:= X_{N_k}^1 = X_{N_k}^2, \\ R_{k+1} &:= N_{k+1} - N_k, \quad \text{and} \quad R_0 := 0 \end{aligned}$$

and observe that \mathbf{A}_k is the states of the scouts' automata at the time of their k -th meeting, Y_k is their position at this time, and R_k is the time that elapsed from the $k - 1$ -st meeting. The strong Markov property of the scout process then implies that $(Y_k, \mathbf{A}_k, R_k)_{k=0}^\infty$ is a Markov chain on $\mathbb{Z}^2 \times \mathcal{S}^2 \times \mathbb{Z}_{\geq 1}$. Moreover, thanks to the spatial homogeneity of the transition function of $(\mathbf{X}_n, \mathbf{Q}_n)_{n \geq 0}$, we further have for all $k \geq 0$, almost surely

$$\mathbb{P}\left((Y_{k+1} - Y_k, \mathbf{A}_{k+1}, R_{k+1}) \in \cdot \mid (Y_m, \mathbf{A}_m, R_m)_{m=0}^k\right) = \mathbb{P}\left((Y_{k+1} - Y_k, \mathbf{A}_{k+1}, R_{k+1}) \in \cdot \mid \mathbf{A}_k\right). \quad (3)$$

Notice that R_k is an upper bound on the maximal distance traveled by the scouts between their $k - 1$ -st and k -th meetings, namely,

$$\forall i = 1, 2, n \in [N_{k-1}, N_k] : \max\{\|X_n^i - Y_{k-1}\|, \|X_n^i - Y_k\|\} \leq R_k. \quad (4)$$

In particular,

$$\inf\{k \geq 0 : \|Y_k - x\| \leq R_{k+1}\} \leq \inf\{n \geq 0 : X_n^1 = x \text{ or } X_n^2 = x\}.$$

It follows that if (1) is false, then for every $x \in \mathbb{Z}^2$, we must have

$$\mathbb{E}(\inf\{k \geq 0 : \|Y_k - x\| \leq R_{k+1}\}) < \infty. \quad (5)$$

The process $(Y_k, \mathbf{A}_k, R_k)_{k \geq 0}$ can be viewed as describing a single *explorer* on \mathbb{Z}^2 which has the ability to explore a ball of radius R_{k+1} around its location Y_k for $k = 0, 1, \dots$. Let us first define such a process in a formal manner.

Definition 2.3. Let \mathcal{S} be a finite non-empty state space, $\Pi : \mathcal{S} \times (\mathcal{S} \times \mathbb{Z}^2 \times \mathbb{Z}_{\geq 1}) \rightarrow [0, 1]$ a (probability) transition function, $x_0 \in \mathbb{Z}^2$, and $q_0 \in \mathcal{S}$. An *explorer process* on \mathbb{Z}^2 with state space \mathcal{S} , transition function Π , initial position x_0 and initial state q_0 is a random process $(X_n, Q_n, R_n)_{n \geq 0}$ on $\mathbb{Z}^2 \times \mathcal{S} \times \mathbb{Z}_{\geq 1}$ satisfying

$$(X_0, Q_0, R_0) = (x_0, q_0, 0)$$

and for all $n \geq 1$, $\xi \in \mathbb{Z}^2$, $q \in \mathcal{S}$, $r \in \mathbb{Z}_{\geq 1}$,

$$\mathbb{P}(X_{n+1} - X_n = \xi, Q_{n+1} = q, R_{n+1} = r \mid (X_m, Q_m, R_m)_{m=0}^n) = \Pi(Q_n, (q, \xi, r)) \text{ a.s.}$$

By (3), this definition applies to the process $(Y_k, \mathbf{A}_k, R_k)_{k \geq 0}$. Furthermore, by Proposition 2.2 and (4), the conditional distributions of both $Y_{k+1} - Y_k$ and R_{k+1} given \mathbf{A}_k have (at least) a stretched-exponentially decaying upper tail (with deterministic constants). Proposition 2.4 will now show that if, in addition, (5) holds, then such an explorer process must eventually get trapped in a finite set of grid points.

⁷Unless stated otherwise, the notation $\|\cdot\|$ is used to denote the ℓ_∞ norm.

Proposition 2.4. *Let $(X_n, Q_n, R_n)_{n \geq 0}$ be an explorer process on \mathbb{Z}^2 such that for some $\delta > 0$ and all $n \geq 0$, $u \geq 0$,*

$$\mathbb{P}(\|X_{n+1} - X_n\| + R_{n+1} > u \mid Q_n) \leq \frac{1}{\delta} e^{-\delta \sqrt{u}} \text{ a.s.} \quad (6)$$

If for all $x \in \mathbb{Z}^2$,

$$\mathbb{E}(\inf\{n \geq 0 : \|X_n - x\| \leq R_{n+1}\}) < \infty, \quad (7)$$

there must exist a stopping time τ (for the explorer process) and a non-random $r < \infty$ such that

$$\mathbb{P}(\tau < \infty, \forall n \geq \tau : \|X_n - X_\tau\| < r) = 1. \quad (8)$$

Now, we set $T := N_\tau$ and observe that T is a stopping time for the process $(\mathbf{X}_n, \mathbf{Q}_n)_{n \geq 0}$. Therefore, Proposition 2.2, Proposition 2.4 and the assumed validity of (5) imply that there exist $r < \infty$ and a stopping time T , such that with probability one, after time $T < \infty$, the two scouts meet only inside a ball of radius r around the grid point $X_T^1 = X_T^2 = Y_\tau$. Moreover, they keep meeting infinitely often and the times between successive meetings have finite means. Since this implies, in particular, that whenever a scout is away from this ball, it evolves independently of the other scout, it follows that each scout must always be in a grid point from which the mean time of returning to the ball, *as a single scout process*, is finite. Proposition 2.5 will show that the set of such grid points is a “small” subset of the whole grid.

To make things precise, given $\hat{\alpha} \in \mathbb{R}^2$ with $\|\hat{\alpha}\|_2 = 1$ and $M > 0$, let us define the *thick ray* of width M in direction $\hat{\alpha}$ as

$$\mathcal{R}(\hat{\alpha}, M) := \left\{ x \in \mathbb{R}^2 : |\langle x, \hat{\alpha}^\perp \rangle| < M \text{ and } \langle x, \hat{\alpha} \rangle > -M \right\},$$

where $\hat{\alpha}^\perp$ denotes (any) unit vector which is perpendicular to $\hat{\alpha}$ and $\langle x, y \rangle$ denotes the scalar product between x and y . For what follows, if $(Z_n)_{n \geq 0}$ is a Markov chain on \mathcal{A} and $z_0 \in \mathcal{A}$, then we shall write $\mathbb{P}_{z_0}(\cdot)$ for the probability measure under which $Z_0 = z_0$. The same notation will apply to the corresponding expectation.

Proposition 2.5. *Let $(X_n, Q_n)_{n \geq 0}$ be a single scout process on \mathbb{Z}^2 with state space \mathcal{S} and transition function Π and let also $r > 0$. There exist $m \in \mathbb{Z}_{\geq 1}$, unit vectors $\hat{\alpha}_1, \dots, \hat{\alpha}_m \in \mathbb{R}^2$ and $M < \infty$, such that if $(x_0, q_0) \in \mathbb{Z}^2 \times \mathcal{S}$ are such that*

$$\mathbb{E}_{(x_0, q_0)}(\inf\{n \geq 0 : \|X_n\| < r\}) < \infty \quad (9)$$

then

$$x_0 \in \mathcal{D} := \bigcup_{i=1}^m \mathcal{R}(\hat{\alpha}_i, M). \quad (10)$$

We can now finish the proof of the theorem. Let $(\tilde{X}_n, \tilde{Q}_n)_{n \geq 0}$ be a single scout process on \mathbb{Z}^2 with state space \mathcal{S} and transition function Π as in the conditions of Theorem 2.1. Let also r be

given by Proposition 2.4 applied to the process $(Y_k, \mathbf{A}_k, R_k)_{k \geq 0}$. Applying Proposition 2.5 with $(\tilde{X}_n, \tilde{Q}_n)_{n \geq 0}$ and r , we obtain a subset $\mathcal{D} \subseteq \mathbb{R}^2$ as defined in (10). We now claim that for $i = 1, 2$,

$$\mathbb{P}(X_{T+n}^i \in Y_\tau + \mathcal{D}, \forall n \geq 0) = 1.^8 \quad (11)$$

Indeed, for $n \geq 0$ consider the first time after $T + n$ such that scout i comes to within distance smaller than r of Y_τ , namely,

$$\hat{N}_n^i := \inf \{m \geq 0 : \|X_{T+n+m}^i - Y_\tau\| < r\}.$$

Since $N_{\tau+n} \geq T + n$ and $\|X_{N_{\tau+n}}^i - Y_\tau\| < r$, it follows that $T + n + \hat{N}_n^i \leq N_{\tau+n}$. Using the strong Markov property, the almost surely finiteness of τ and Proposition 2.2 (or (6)), we then have

$$\begin{aligned} \mathbb{E} \hat{N}_n^i &\leq \mathbb{E}(N_{\tau+n} - N_\tau - n) \leq \sum_{k=1}^n \mathbb{E}(\mathbb{E}(N_{\tau+k} - N_{\tau+k-1} \mid \mathbf{Q}_{N_{\tau+k-1}})) \\ &= \sum_{k=1}^n \mathbb{E}(\mathbb{E}_{(0, \mathbf{Q}_{N_{\tau+k-1}})} N_1) < \infty, \end{aligned}$$

where the inner expectation in the last line is with respect to the process $(\mathbf{X}_n, \mathbf{Q}_n)_{n \geq 0}$.

On the other hand,

$$\mathbb{E} \hat{N}_n^i \geq \mathbb{P}(X_{T+n}^i \notin Y_\tau + \mathcal{D}) \cdot \mathbb{E}(\hat{N}_n^i \mid X_{T+n}^i \notin Y_\tau + \mathcal{D}).$$

In light of Proposition 2.4, we know that $\hat{N}_n^i \leq \inf\{m \geq 0 : X_{T+n+m}^i = X_{T+n+m}^{3-i}\}$ with probability one. From this fact and the strong Markov property, it follows that

$$\begin{aligned} &\mathbb{E}(\hat{N}_n^i \mid X_{T+n}^i \notin Y_\tau + \mathcal{D}) \\ &= \mathbb{E}\left(\left(\mathbb{E}_{(X_{T+n}^i, Q_{T+n}^i)} \inf\{m \geq 0 : \|\tilde{X}_m - Y_\tau\| < r\}\right) \mid X_{T+n}^i \notin Y_\tau + \mathcal{D}\right), \end{aligned}$$

where the inner expectation is just with respect to the process $(\tilde{X}_n, \tilde{Q}_n)_{n \geq 0}$. But by Proposition 2.5, under the conditioning the inner expectation is infinite almost surely. We therefore must have that $\mathbb{P}(X_{T+n}^i \notin Y_\tau + \mathcal{D}) = 0$. Summing over all n and using the union bound yields (11).

Finally, the validity of (11) for $i = 1, 2$ implies that with probability one,

$$X_n^i \in B_T \cup (Y_\tau + \mathcal{D}) \quad \forall i = 1, 2 \text{ and } n = 0, 1, \dots,$$

where henceforth B_ρ is the closed ball of radius ρ around 0 in the ℓ^∞ norm. Since $T < \infty$, the random set $B_T \cup (Y_\tau + \mathcal{D})$ is always a strict subset of \mathbb{Z}^2 . It follows that there must exist $x \in \mathbb{Z}^2$ such that with positive probability, x will not be reached by either scout. This in turn shows that (1) holds, in contradiction to what we have assumed in the first place, thus establishing Theorem 2.1.

⁸If $x \in \mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$ then $x + A$ stands for the set $\{x + y : y \in A\}$.

2.2 Single and Generalized Scout Processes and Underlying Automaton

If $(X_n, Q_n)_{n \geq 0}$ is a single scout process, its environment vector E_n is always 0^S and hence one can effectively consider a reduced version $\Pi' : \mathcal{S} \times (\mathcal{S} \times \{-1, 0, +1\}^d) \mapsto [0, 1]$ of the transition function Π of the underlying protocol, where

$$\Pi'(q, (q', \xi)) = \Pi((q, 0^S), (q', \xi)) \text{ , } q, q' \in \mathcal{S} \text{ , } \xi \in \{-1, 0, +1\}^d \text{ .}$$

The marginal process $(Q_n)_{n \geq 0}$ then becomes a Markov chain on \mathcal{S} by itself (with transition function given by $(q, q') \mapsto \sum_{\xi} \Pi'(q, (q', \xi))$) and we shall refer to this process as the scout's (underlying) *automaton*. In particular, the *irreducible classes* of the automaton will be the irreducible states classes of the Markov chain $(Q_n)_{n \geq 0}$ and the automaton will be called *irreducible* in this Markov chain is such.

In the sequel, we shall also need to consider a (single) *generalized scout process* $(\hat{X}_n, \hat{Q}_n)_{n \geq 0}$, which is defined as the single scout process above, but with more general steps. Formally, for $d \geq 1$, the process, which now takes values in $\mathbb{R}^d \times \mathcal{S}$, is defined exactly as before, only that the transition function now takes the form $\hat{\Pi} : \mathcal{S} \times (\mathcal{S} \times \mathcal{W}) \mapsto [0, 1]$, where \mathcal{W} is some discrete nonempty subset of \mathbb{R}^d . As in the case of a single scout process, the process $(\hat{Q}_n)_{n \geq 0}$, which is Markovian by itself, will be referred to as the underlying automaton. The following lemma will be used more than once in what follows.

Lemma 2.6. *Let $(\hat{X}_n, \hat{Q}_n)_{n \geq 0}$ be a generalized single scout process on \mathbb{R}^d for $d \geq 1$ with state space \mathcal{S} , transition function $\hat{\Pi}$, initial position $x_0 = 0$ and some initial state $q_0 \in \mathcal{S}$. Suppose also that its automaton is irreducible and set*

$$T := \inf\{n > 0 : \hat{Q}_n = q_0\} \text{ , } \zeta := \hat{X}_T \text{ .}$$

If $\mathbb{P}(\zeta = 0) = 1$, then there exist $r \in (0, \infty)$ and a finite subset $\mathcal{A} \subseteq B_r \subset \mathbb{R}^d$ such that

$$\mathbb{P}(\forall n \geq 0 : \hat{X}_n \in \mathcal{A}) = 1 \text{ .} \tag{12}$$

and for all $x \in \mathcal{A}$ and $u \in \mathbb{Z}_{\geq 0}$,

$$\mathbb{P}(\forall 0 \leq n \leq u : \hat{X}_n \neq x) \leq C e^{-C' u} \text{ .} \tag{13}$$

with some $C, C' > 0$.

Proof. We will show that for each $q \in \mathcal{S}$ there exists $x_q \in \mathbb{R}^d$ such that almost surely for all n ,

$$\hat{Q}_n = q \Rightarrow \hat{X}_n = x_q \text{ .} \tag{14}$$

This will prove (12) with $\mathcal{A} := \{x_q : q \in \mathcal{S}\}$ and $r = \max\{\|x_q\| : q \in \mathcal{S}\}$. To this end, observe first that if $\zeta = 0$, then with probability one

$$\hat{Q}_n = q_0 \Rightarrow \hat{X}_n = 0 \text{ .} \tag{15}$$

Suppose now that there are $x_q \neq x'_q \in \mathbb{Z}^d$ and $n_1, n_2 \geq 0$ such that

$$\mathbb{P}(\widehat{Q}_{n_1} = q, \widehat{X}_{n_1} = x_q) > 0, \quad \mathbb{P}(\widehat{Q}_{n_2} = q, \widehat{X}_{n_2} = x'_q) > 0. \quad (16)$$

Since $(\widehat{Q}_n)_{n \geq 0}$ is irreducible and hence recurrent, it follows from (15), (16) and the Markov property that there exist $m_1, m_2 \geq 0$ such that

$$\mathbb{P}_{(x_q, q)}(\widehat{Q}_{m_1} = q_0, \widehat{X}_{m_1} = 0) > 0, \quad \mathbb{P}_{(x'_q, q)}(\widehat{Q}_{m_2} = q_0, \widehat{X}_{m_2} = 0) > 0. \quad (17)$$

But then using the product rule with the first event in (16) and the second event in (17) and the Markov property again, we get

$$\mathbb{P}(\widehat{Q}_{n_1+m_2} = q_0, \widehat{X}_{n_1+m_2} = x_q - x'_q) > 0.$$

Since $x_q - x'_q \neq 0$, this leads to a contradiction to (15) occurring with probability 1.

To show (13), we use the irreducibility and finiteness of the Markov chain $(\widehat{Q}_n)_{n \geq 0}$ together with standard theory to claim that the hitting time of any $q \in \mathcal{S}$ has an exponentially decaying upper tail. Then (13) follows by virtue of (14). □

2.3 Two Scouts on \mathbb{Z}^2 Must Meet Frequently

In this section we prove Proposition 2.2. We shall do this gradually, first assuming that the two scouts evolve independently as single scout processes and have irreducible automata (in the sense discussed in beginning of Subsection 2.2). We shall then remove the irreducibility restriction and finally consider the full (dependent) two-scout process.

Let therefore $(X_n^1, Q_n^1)_{n \geq 0}$ and $(X_n^2, Q_n^2)_{n \geq 0}$ be two independent single scout processes. For $i = 1, 2$, suppose that scout process i has state space \mathcal{S}^i , transition function Π^i , initial position $x_0^i \in \mathbb{Z}^2$ and initial state $q_0^i \in \mathcal{S}^i$. As before, we shall write \mathbf{X}_n for (X_n^1, X_n^2) , \mathbf{Q}_n for (Q_n^1, Q_n^2) , etc. We also define,

$$N := \inf\{n > 0 : X_n^1 = X_n^2\}$$

and for $y \in \mathbb{Z}^2$,

$$\tau_y := \inf\{n \geq 0 : X_n^1 = y \text{ or } X_n^2 = y\}.$$

The first lemma deals with the case of irreducible automata.

Lemma 2.7. *Let (X_n^1, Q_n^1) and (X_n^2, Q_n^2) be two independent single scout processes on \mathbb{Z}^2 with state spaces \mathcal{S} and transition functions Π , starting from initial positions \mathbf{x}_0 and initial states \mathbf{q}_0 . Suppose also that the automaton of each scout processes is irreducible. Then either*

$$|\{y \in \mathbb{Z}^2 : \mathbb{E}(N \wedge \tau_y) = \infty\}| = \infty \quad (18)$$

or for all $u \geq 0$,

$$\mathbb{P}(N > u) \leq \frac{1}{\delta} e^{-\delta(\sqrt{u} - \|x_0^1 - x_0^2\|)}. \quad (19)$$

for some $\delta > 0$ which depends only on the transition functions $\mathbf{\Pi}$.

Proof. We shall observe each scout i at times of successive returns to q_0^i as well as both scouts together at times of successive simultaneous returns to states $\mathbf{q}_0 = (q_0^1, q_0^2)$. This will turn the scouts processes into random walks.

Formally, let $T_0^i = 0$, $T_0^\Delta = 0$ and for $k = 1, \dots$ define:

$$\begin{aligned} T_k^i &:= \inf\{n \geq T_{k-1}^i : Q_n^i = q_0^i\} \quad i = 1, 2, \\ T_k^\Delta &:= \inf\{n \geq T_{k-1}^\Delta : Q_n^1 = q_0^1, Q_n^2 = q_0^2\}. \end{aligned}$$

For $n \geq 0$, and with $\mathbf{e}_1 = (1, 0) \in \mathbb{Z}^2$, we also set:

$$S_n^i := \langle X_{T_n^i}^i, \mathbf{e}_1 \rangle \quad i = 1, 2, \quad S_n^\Delta := \langle X_{T_n^\Delta}^1 - X_{T_n^\Delta}^2, \mathbf{e}_1 \rangle.$$

Then for $i = 1, 2, \Delta$ and $k \geq 1$ we let:

$$s_0^i := S_0^i, \quad \zeta_k^i := S_k^i - S_{k-1}^i, \quad \nu_k^i := T_k^i - T_{k-1}^i, \quad R_k^i = 2\nu_k^i.$$

For $i = 1, 2$, the Markov property and spatial homogeneity of the process (X_n^i, Q_n^i) imply that the triplets $(\zeta_k^i, \nu_k^i, R_k^i)_{k \geq 1}$ are i.i.d. and consequently that $(S_n^i, T_n^i)_{n \geq 0}$ is a random walk. Since the process $(\mathbf{X}_n, \mathbf{Q}_n)_{n \geq 0}$ can be viewed as a single scout process on \mathbb{Z}^4 , the same applies to $(\zeta_k^\Delta, \nu_k^\Delta, R_k^\Delta)_{k \geq 1}$ and $(S_n^\Delta, T_n^\Delta)_{n \geq 0}$. Moreover, since the underlying scout automata are irreducible, standard Markov chain theory implies that there exists $C > 0$ such that for $i = 1, 2, \Delta$,

$$\mathbb{P}(\nu_1^i > u) \leq C^{-1} e^{-Cu}. \quad (20)$$

Then, since $|\zeta_1^i| \leq R_1^i = 2\nu_1^i$, we also have

$$\mathbb{P}(|\zeta_1^i| + \nu_1^i + R_1^i > u) \leq \delta^i e^{-\delta^i u}. \quad (21)$$

for some $\delta^i > 0$.

For $i = 1, 2, \Delta$, let the *effective* drift of walk i be defined as

$$d^i := (\mathbb{E}\zeta_1^i) / (\mathbb{E}\nu_1^i).$$

We next claim that

$$d^\Delta = d^1 - d^2.$$

Setting $X_n^\Delta := X_n^1 - X_n^2$, it is enough to show that for $i = 1, 2, \Delta$, with probability one,

$$\lim_{m \rightarrow \infty} \langle X_m^i, \mathbf{e}_1 \rangle / m = d^i. \quad (22)$$

To this end, for all $m \geq 0$, we set

$$K_m^i := \sup\{n \geq 0 : T_n^i \leq m\}.$$

By the law of large numbers, with probability one, for all $\epsilon > 0$, there exists n_0 such that for all $n > n_0$,

$$n(\mathbb{E}\zeta_1^i - \epsilon) \leq S_n^i \leq n(\mathbb{E}\zeta_1^i + \epsilon) \quad (23)$$

and

$$n(\mathbb{E}\nu_1^i - \epsilon) \leq T_n^i \leq n(\mathbb{E}\nu_1^i + \epsilon). \quad (24)$$

Since $m \in [T_{K_m^i}^i, T_{K_m^i+1}^i)$ by definition, for all m large enough (24) implies

$$m/(\mathbb{E}\nu_1^i + 2\epsilon) \leq K_m^i \leq m/(\mathbb{E}\nu_1^i - \epsilon). \quad (25)$$

But then from (23),

$$m \frac{\mathbb{E}\zeta_1^i - \epsilon}{\mathbb{E}\nu_1^i + 2\epsilon} \leq S_{K_m^i}^i \leq m \frac{\mathbb{E}\zeta_1^i + \epsilon}{\mathbb{E}\nu_1^i - \epsilon}.$$

Since this is true for all $\epsilon > 0$, this shows that as $m \rightarrow \infty$, almost-surely

$$S_{K_m^i}^i/m \rightarrow (\mathbb{E}\zeta_1^i)/(\mathbb{E}\nu_1^i) = d^i. \quad (26)$$

At the same time,

$$|\langle X_m^i, e_1 \rangle - S_{K_m^i}^i| \leq T_{K_m^i+1}^i - T_{K_m^i}^i.$$

Dividing by m both sides above, taking $m \rightarrow \infty$ and using (24) and (25) we obtain

$$\lim_{m \rightarrow \infty} |\langle X_m^i, e_1 \rangle/m - S_{K_m^i}^i/m| = 0.$$

Together with (26) this shows (22).

Care must be taken to handle the possible degeneracy of the steps of the walks $(S_n^i, T_n^i)_{n \geq 0}$, for $i = 1, 2, \Delta$. We therefore first assume that

$$\mathbb{P}(\zeta_1^\Delta = 0) < 1, \quad (27)$$

and relegate the treatment of the complementary case to a later point in the proof. As for $i = 1, 2$, if ζ_1^i/ν_1^i is a non-random constant, then we must have $\zeta_1^i = d^i \nu_1^i$ with probability 1. Applying then Lemma 2.6 for the (single, irreducible one dimensional) generalized scout process $(\hat{X}_n^i, \hat{Q}_n^i)_{n \geq 0}$, defined by setting

$$\hat{X}_n^i := \langle X_n^i - x_0^i, e_1 \rangle - d^i n, \quad \hat{Q}_n^i := Q_n^i, \quad ; \quad n = 0, \dots, \quad (28)$$

we obtain $r^i \in (0, \infty)$ such that almost surely, $|\hat{X}_n^i| \leq r^i$ for all $n \geq 0$. It follows that

$$\mathbb{P}(\forall n \geq 0 : |\langle X_n^i - x_0^i, e_1 \rangle - d^i n| \leq r^i) = 1, \quad (29)$$

and accordingly we redefine $(\zeta_k^i, \nu_k^i, R_k^i)_{k \geq 1}$ and $(S_n^i, T_n^i)_{n \geq 0}$ as

$$\forall k \geq 1 : \zeta_k^i := d^i, \nu_k^i = 1, R_k^i = r^i \quad ; \quad \forall n \geq 0 : S_n^i := s_0^i + d^i n, T_n^i = n. \quad (30)$$

Observe that under the new definition, $(S_n^i, T_n^i)_{n \geq 0}$ is still a random walk, albeit with deterministic steps which trivially satisfy (21) for some $\delta^i > 0$. Moreover $K_m^i = m$.

With the above definitions, the process $(S_n^\Delta, R_n^\Delta)_{n \geq 0}$ is a look-around random walk of the type considered in sub-section 3.1 and the processes $(S_n^i, T_n^i, R_n^i)_{n \geq 0}$ for $i = 1, 2$, form a pair of time-varying look-around random-walks of the type considered in subsection 3.2. Moreover, by the definition of R_n^i , we have

$$\forall i = 1, 2, m \geq 0 : |\langle X_m^i, e_1 \rangle - S_{K_m^i}^i| \leq R_{K_m^i+1}^i. \quad (31)$$

We now appeal to Proposition 3.8 with the random walks $(S_n^1, T_n^1, R_n^1)_{n \geq 0}$ and $(S_n^2, T_n^2, R_n^2)_{n \geq 0}$ and to Lemma 3.6 with the random walk $(S_n^\Delta, R_n^\Delta)_{n \geq 0}$. Let $\rho > 0$ be as given by the first proposition, τ_ρ be as in (67) for the walk S_n^Δ and set $N_\rho^\Delta := T_{\tau_\rho}^\Delta$. Suppose first that $\mathbb{P}(N_\rho^\Delta < \infty, N_\rho^\Delta < N) > 0$. Then there exist $n \geq 0$ and $s_1^i \in \mathbb{Z}$ for $i = 1, 2, \Delta$ such that (70) holds and

$$\mathbb{P}(N > N_\rho^\Delta = n, S_n^i = s_1^i : i = 1, 2, \Delta) > 0. \quad (32)$$

For such s_1^i , by Proposition 3.8, there exist $-\infty < x < y < \infty$ such that $|y - x| > 2$ and (71) holds. But then for each $z \in \mathbb{Z}^2$ satisfying $\langle z, e_1 \rangle \in [x, y]$ and $\|z - x_0^1\| > n$ and $\|z - x_0^2\| > n$, by the Markov property, the fact that N and N_ρ^Δ are stopping times and (31),

$$\begin{aligned} \mathbb{E}(N \wedge \tau_z \mid N > N_\rho^\Delta = n, S_n^i = s_1^i : i = 1, 2, \Delta) \\ \geq n + \mathbb{E}_{(s_1^1, s_1^2)}(\min\{\sigma, \tau_{[x,y]}^1, \tau_{[x,y]}^2\}) = \infty, \end{aligned}$$

where $\tau_{[x,y]}^i$ is defined as in (69). In light of (32) shows (18).

Otherwise, $\mathbb{P}(N \leq N_\rho^\Delta) = 1$ and then by Lemma 3.6 (which is in force due to (27)) and (20), for all $u \geq 0$

$$\begin{aligned} \mathbb{P}(N > u) &\leq \mathbb{P}(N_\rho^\Delta > u) \\ &\leq \mathbb{P}_{s_0^\Delta}(\tau_\rho^\Delta > \sqrt{u}) + \sum_{m=1}^{\lfloor \sqrt{u} \rfloor} \mathbb{P}(\nu_m^\Delta > \sqrt{u}) \\ &\leq \delta^{-1} e^{-\delta(\sqrt{u} - |s_0^\Delta|)} + C \sqrt{u} e^{-C' \sqrt{u}} \leq \frac{1}{\delta'} e^{-\delta'(\sqrt{u} - \|x_0^1 - x_0^2\|)}, \end{aligned}$$

for some $C, C' > 0$. This shows (19).

It remains to treat the case when (27) does not hold. In this case, we first replace $e_1 = (1, 0)$ by $e_2 := (0, 1)$ in the entire argument. If (27) now holds, then we proceed as before and the proof is complete. If not, then we must have $(X_{T_1^\Delta}^1 - X_{T_1^\Delta}^2) - (x_0^1 - x_0^2) = 0$ with probability 1. But then, by Lemma 2.6 applied to the generalized scout process

$$(\hat{X}_n, \hat{Q}_n)_{n \geq 0} := ((X_n^1 - X_n^2) - (x_0^1 - x_0^2), (Q_n^1, Q_n^2))_{n \geq 0}, \quad (33)$$

there exists $\mathcal{A} \subseteq \mathbb{Z}^2$ such that (12) and (13) hold.

If $-(x_0^1 - x_0^2) \in \mathcal{A}$, then by (13) we have $\mathbb{P}(N > u) \leq Ce^{-C'u}$ for all $u \geq 0$, which in particular shows (19). If not, then $N = \infty$ with probability one and hence almost surely

$$\forall z \in \mathbb{Z}^2 : N \wedge \tau_z = \tau_z. \quad (34)$$

Thanks to Lemma 3.1 or Lemma 3.3 (depending on whether $d^1 \neq 0$ or $d^1 = 0$) applied to the look-around random walk $(S_n^1, R_n^1)_{n \geq 0}$, as defined above, there exist $x^1 \in \mathbb{R}$, $\alpha^1 \in \{-1, +1\}$ and $C > 0$ such that for all $z \in \mathbb{Z}^2$ with $\alpha^1(\langle z, e_1 \rangle - x^1) > 0$ and $u \geq 1$

$$\mathbb{P}(\forall n \leq u : X_n^1 \neq z) \geq \mathbb{P}(\forall n \leq u : |S_n^1 - \langle z, e_1 \rangle| > R_{n+1}^1) \geq C/\sqrt{u}. \quad (35)$$

By a similar argument, there exists there exist $x^2 \in \mathbb{R}$, $\alpha^2 \in \{-1, +1\}$ and $C' > 0$ such that for all $z \in \mathbb{Z}^2$ with $\alpha^2(\langle z, e_2 \rangle - x^2) > 0$ and $u \geq 1$,

$$\mathbb{P}(\forall n \leq u : X_n^2 \neq z) \geq C'/\sqrt{u}. \quad (36)$$

It follows then by the independence of the scout processes that for all $z \in \mathbb{Z}^2$ satisfying $\alpha^1(\langle z, e_1 \rangle - x^1) > 0$ and $\alpha^2(\langle z, e_2 \rangle - x^2) > 0$ and $u \geq 1$,

$$\mathbb{P}(\tau_z > u) \geq CC'/u. \quad (37)$$

The tail formula for expectation then gives $\mathbb{E}\tau_z = \infty$ for all such z . Since there infinitely many such z -s and thanks to (34), we obtain (18). \square

Next, we remove the restriction to irreducible automata.

Lemma 2.8. *Let (X_n^1, Q_n^1) and (X_n^2, Q_n^2) be two independent scout processes on \mathbb{Z}^2 with state spaces \mathcal{S} and transition functions Π , starting from $x_0^1 = x_0^2 = 0$ and initial states \mathbf{q}_0 . Then either*

$$|\{y \in \mathbb{Z}^2 : \mathbb{E}(N \wedge \tau_y) = \infty\}| = \infty \quad (38)$$

or there exists $\delta > 0$ such that for all $u \geq 0$.

$$\mathbb{P}(N > u) \leq \frac{1}{\delta} e^{-\delta\sqrt{u}}. \quad (39)$$

Proof. For $i = 1, 2$, let $\mathcal{S}_1^i, \dots, \mathcal{S}_{m^i}^i \subset \mathcal{S}^i$ for some $m^i \geq 1$ be the recurrent irreducible classes of automaton $(Q_n^i)_{n \geq 0}$. Note that if $q_0^i \in \mathcal{S}_l^i$ for some $l \in \{1, \dots, m^i\}$ then (X_n^i, Q_n^i) is also a single scout process with state space \mathcal{S}_l^i and transition function Π_l^i , which is the proper restriction of Π^i to \mathcal{S}_l^i . Moreover, its automaton then is irreducible. Define,

$$\sigma^i := \inf\{n \geq 0 : Q_n^i \in \cup_{l=1}^{m^i} \mathcal{S}_l^i\} \quad , i = 1, 2,$$

and set $\sigma := \sigma^1 \vee \sigma^2$. By standard Markov chain theory,

$$\mathbb{P}(\sigma^i > u) \leq C^{-1} e^{-Cu} \quad , \quad i = 1, 2,$$

which implies the same for σ . In particular σ, σ^1 and σ^2 are finite almost surely.

Suppose first that there exist $n \geq 0$, l^1, l^2 , $\mathbf{x} \in \mathbb{Z}^4$ and $\mathbf{q} \in \mathcal{S}_{l^1}^1 \times \mathcal{S}_{l^2}^2$ such that

$$\mathbb{P}(N > \sigma = n, \mathbf{X}_n = \mathbf{x}, \mathbf{Q}_n = \mathbf{q}) > 0 \quad (40)$$

and that (18) holds in Lemma 2.7 applied to $(\mathbf{X}_n, \mathbf{Q}_n)_{n \geq 0}$ as two independent scout processes with transition functions $\Pi_{l^1}^1, \Pi_{l^2}^2$ respectively and with $\mathbf{x}_0 = \mathbf{x}$ and $\mathbf{q}_0 = \mathbf{q}$. Since N and σ are stopping times, it then follows that for any $y \in \mathbb{Z}^2$ with $\|y\| > n$,

$$\mathbb{E}(N \wedge \tau_y \mid N > \sigma = n, \mathbf{X}_n = \mathbf{x}, \mathbf{Q}_n = \mathbf{q}) = n + \mathbb{E}_{(\mathbf{x}, \mathbf{q})}(N \wedge \tau_y) = \infty, \quad (41)$$

Together with (41) this gives (38).

Otherwise, by Lemma 2.7, anytime (40) holds, we must also have (19) with some $\delta^{(l^1, l^2)} > 0$, which depends only the transition functions Π^{l^1} and Π^{l^2} . Setting

$$\delta_0 = \min\{\delta^{(l^1, l^2)} : l^1 = 1, \dots, m^1, l^2 = 1, \dots, m^2\}$$

we then have with some $C > 0$,

$$\begin{aligned} \mathbb{P}(N > u) &\leq \mathbb{P}(N > u, \sigma \leq \sqrt{u}/4) + \mathbb{P}(\sigma > \sqrt{u}/4) \\ &\leq \mathbb{E}\left(\mathbb{P}_{(\mathbf{X}_\sigma, \mathbf{Q}_\sigma)}(N > u) 1_{\{\sigma \leq \sqrt{u}/4\}}\right) + C^{-1}e^{-C\sqrt{u}} \\ &\leq \delta_0^{-1}e^{-\delta_0(\sqrt{u}-\sqrt{u}/2)} + C^{-1}e^{-C\sqrt{u}} \leq \delta^{-1}e^{-\delta\sqrt{u}}, \end{aligned}$$

where we have used the strong Markov property and the fact that $\|X_\sigma^1 - X_\sigma^2\| \leq 2\sigma$. This shows (39). \square

Proof of Proposition 2.2. The proof follows by induction on k . For $k = 1$, since up to time N , the two-scout process evolves as two independent single scout processes, if y and \mathbf{q}_0 are such that $\mathbb{E}_{(\mathbf{0}, \mathbf{q}_0)}(N \wedge \tau_y) = \infty$ for two independent scouts, then also

$$\mathbb{E}_{(\mathbf{0}, \mathbf{q}_0)}(\tau_y) \geq \mathbb{E}_{(\mathbf{0}, \mathbf{q}_0)}(N \wedge \tau_y) = \infty, \quad (42)$$

for the two-scout process. Lemma 2.8 with $(\mathcal{S}^1, \Pi^1) = (\mathcal{S}^2, \Pi^2) = (\mathcal{S}, \Pi)$, and the assumption that (1) does not hold, imply then that \mathbf{q}_0 must be such that (39) holds. Since $N_1 = N$, this gives the case $k = 1$.

Suppose now that (2) holds up to $k - 1$. In particular, this shows that $N_{k-1} < \infty$ almost surely. Conditioning on $\mathcal{F}_{N_{k-1}}$,⁹ the strong Markov property and the spatial homogeneity of the underlying processes imply that almost surely,

$$\mathbb{P}(N_k - N_{k-1} \in \cdot \mid \mathcal{F}_{N_{k-1}}) = \mathbb{P}_{(\mathbf{0}, \mathbf{Q}_{N_{k-1}})}(N \in \cdot).$$

⁹Throughout, for a stopping time Z , we use the standard notation \mathcal{F}_Z to denote the sigma-algebra generated by Z .

If with positive probability $\mathbf{Q}_{N_{k-1}} = \mathbf{q}_0$ for \mathbf{q}_0 such that (38) holds, then since the number of vertices visited by both scouts up to time N_{k-1} is finite, it follows as in (42), that (1) cannot hold. We therefore must have for all $u \geq 0$,

$$\mathbb{P}(N_k - N_{k-1} > u \mid \mathcal{F}_{N_{k-1}}) \leq \delta^{-1} e^{-\delta\sqrt{u}} \text{ a.s.}$$

The tower property for conditional expectation shows (2) . □

2.4 One Explorer Must Eventually Get Trapped

In this section we prove Proposition 2.4. As in the case of a single scout process, if $(X_n, Q_n, R_n)_{n \geq 0}$ is an explorer process with state space \mathcal{S} , then $(Q_n)_{n \geq 0}$ is a Markov chain on \mathcal{S} , to be referred to as the explorer's *automaton*. We begin by assuming that this automaton is irreducible.

Lemma 2.9. *Let $(X_n, Q_n, R_n)_{n \geq 0}$ be an explorer process on \mathbb{Z}^2 with state space \mathcal{S} and transition function Π , starting from $x_0 = 0$ and some $q_0 \in \mathcal{S}$. Suppose also that its automaton is irreducible and that there exists $\delta > 0$ such that for all $q \in \mathcal{S}$, $n \geq 0$ and $u \geq 0$,*

$$\mathbb{P}(\|X_{n+1} - X_n\| + R_{n+1} > u \mid Q_n) \leq \frac{1}{\delta} e^{-\delta\sqrt{u}}. \quad (43)$$

If it holds that

$$|\{x \in \mathbb{Z}^2 : \mathbb{E}(\inf\{n \geq 0 : \|X_n - x\| \leq R_{n+1}\}) = \infty\}| < \infty, \quad (44)$$

then there must exist $r > 0$, which depends only on Π , such that

$$\mathbb{P}(\forall n \geq 0 : \|X_n\| < r) = 1.$$

Proof. Let $T_0 = 0$ and for $k = 1, \dots$, set:

$$\begin{aligned} T_k &:= \inf\{n > T_{k-1} : Q_n = q_0\} \\ \zeta_k &:= X_{T_k} - X_{T_{k-1}} \\ \nu_k &:= T_k - T_{k-1} \\ R'_k &:= \sum_{n=T_{k-1}}^{T_k-1} R_{n+1}. \end{aligned} \quad (45)$$

Since the Markov chain $(Q_n)_{n \geq 0}$ is irreducible and hence recurrent, all times T_k are finite almost surely. The Markov property then implies that the triplets $(\zeta_k, \nu_k, R'_k)_{k \geq 1}$ are i.i.d. Moreover, from standard Markov chain theory, there exists $C > 0$ such that for all $u \geq 0$

$$\mathbb{P}(\nu_k > u) \leq C^{-1} e^{-Cu}.$$

In addition (43) and the strong Markov property implies that for all $m \geq 1$ and $u \geq 0$,

$$\mathbb{P}(\|X_{T_{k-1}+m} - X_{T_{k-1}+m-1}\| + R_{T_{k-1}+m} > u) \leq \delta^{-1} e^{-\delta\sqrt{u}}.$$

Then for all $k \geq 1$ and $u \geq 0$, by the union bound,

$$\begin{aligned} \mathbb{P}(\|\zeta_k\| + R'_k > u) &\leq \mathbb{P}(\nu_k \geq \sqrt{u}) + \sum_{m=1}^{\lfloor \sqrt{u} \rfloor} \mathbb{P}(\|X_{T_{k-1}+m} - X_{T_{k-1}+m-1}\| + R_{T_{k-1}+m} > \sqrt{u}) \\ &\leq C\sqrt{u}e^{-C'u^{1/4}} \leq \frac{1}{\delta'}e^{-\delta'u^{1/4}} \end{aligned}$$

for some $\delta' > 0$.

Setting $S_n = \langle X_{T_n}, e_1 \rangle$ for $n \geq 0$, we observe that the process (S_n, R'_n) is of the type handled by Lemma 3.5. Moreover, if (44) holds, then it is also true that

$$|\{x \in \mathbb{Z} : \mathbb{E}(\inf\{n \geq 0 : |S_n - x| < R'_{n+1}\}) = \infty\}| < \infty,$$

Then by Lemma 3.5, we must have $\langle \zeta_1, e_1 \rangle = 0$ with probability one. Repeating the same with $S_n = \langle X_{T_n}, e_2 \rangle$, we get $\langle \zeta_1, e_2 \rangle = 0$. Thus, we conclude that $\zeta_1 = 0$ with probability one. Invoking then Lemma 2.6 for the process $(X_n, Q_n)_{n \geq 0}$, which is, in particular, a generalized scout process, the proof is complete. \square

The proof of Proposition 2.4 is now straightforward,

Proof. Let τ be the first time the Markov chain $(Q_n)_{n \geq 0}$ enters a recurrent class in \mathcal{S} . By standard Markov chain theory τ is finite almost surely. Conditional on \mathcal{F}_τ , by the Markov property, $(X_{\tau+n}, Q_{\tau+n}, R_{\tau+n})_{n \geq 0}$ is an explorer process with an irreducible automaton. If (7) holds then there could be at most τ random vertices $x \in \mathbb{Z}^2$ such that,

$$\mathbb{E}(\inf\{n \geq 0 : \|X_{\tau+n} - x\| \leq R_{\tau+n+1}\} | \mathcal{F}_\tau) = \infty \quad \text{a.s.},$$

Thus by Lemma 2.9, there exists a finite $R \in \mathcal{F}_\tau$ such that,

$$\mathbb{P}(\forall n \geq 0 : \|X_{\tau+n} - X_\tau\| < R | \mathcal{F}_\tau) = 1 \quad \text{a.s.}$$

Since the number of recurrent classes is finite, we may replace R above by a non-random $r > 0$. Taking expectation, we recover (8). \square

2.5 One Trapped Scout Cannot Cover the Whole Space

In this section we prove Proposition 2.5. One more time, we start with the case of an irreducible automaton.

Lemma 2.10. *Let $((X_n, Q_n) : n \geq 1)$ be a single scout process on \mathbb{Z}^2 with state space \mathcal{S} and suppose that its automaton is irreducible. Let also $r < \infty$. There exist $\hat{\alpha} \in \mathbb{R}^2$ with $\|\hat{\alpha}\|_2 = 1$ and $M < \infty$ such that if $x_0 \in \mathbb{R}^2$ and $q_0 \in \mathcal{S}$ are such that*

$$\mathbb{E}_{(x_0, q_0)}(\inf\{n \geq 0 : \|X_n\| < r\}) < \infty \tag{46}$$

then

$$x_0 \in \mathcal{R}(\hat{\alpha}, M).$$

Proof. Let $x_0 \in \mathbb{Z}^2$ and $q_0 \in \mathcal{S}$ and suppose that the scout process starts from position x_0 and state q_0 . Defining T_k , ζ_k and ν_k as in (45), we have that $(\zeta_k, \nu_k)_{k \geq 1}$ are i.i.d. and that

$$\mathbb{P}(\nu_k > u) \leq C^{-1} e^{-Cu}, \quad (47)$$

for some $C > 0$. In particular,

$$S_n := x_0 + \sum_{k=1}^n \zeta_k = X_{T_n}, \quad n = 0, \dots,$$

is a random walk on \mathbb{Z}^2 starting from x_0 . Now define,

$$\hat{\alpha} := \begin{cases} \mathbb{E}(\zeta_1) / \|\mathbb{E}(\zeta_1)\|_2 & \text{if } \mathbb{E}(\zeta_1) \neq 0, \\ 0 & \text{if } \mathbb{E}(\zeta_1) = 0, \end{cases}$$

and let α^\perp be any unit vector which is perpendicular to $\hat{\alpha}$. Setting also for $n, k \geq 0$,

$$\begin{aligned} \zeta_k^\alpha &:= \langle \zeta_k, \hat{\alpha} \rangle, \quad S_n^\alpha := \langle S_n, \hat{\alpha} \rangle, \quad x_0^\alpha := \langle x_0, \hat{\alpha} \rangle, \\ \zeta_k^\perp &:= \langle \zeta_k, \alpha^\perp \rangle, \quad S_n^\perp := \langle S_n, \alpha^\perp \rangle, \quad x_0^\perp := \langle x_0, \alpha^\perp \rangle, \end{aligned}$$

we see that both $(S_n^\alpha)_{n \geq 0}$ and $(S_n^\perp)_{n \geq 0}$ are random walks on \mathbb{R} with steps ζ_k^α and ζ_k^\perp and initial positions x_0^α and x_0^\perp , respectively. Moreover, by definition $\mathbb{E}\zeta_1^\perp = 0$.

If $\mathbb{P}(\zeta_1^\alpha = 0) = 1$, then Lemma 2.6 applied to the generalized scout process $(\langle X_n, \hat{\alpha} \rangle, Q_n)_{n \geq 0}$ implies the existence of $r^\alpha < \infty$ such that $|\langle X_n, \hat{\alpha} \rangle - x_0^\alpha| < r^\alpha$ for all $n \geq 0$ with probability 1. If we therefore set

$$R_n^\alpha := \begin{cases} r + \nu_n & \text{if } \mathbb{P}(\zeta_1^\alpha = 0) < 1, \\ r + r^\alpha & \text{if } \mathbb{P}(\zeta_1^\alpha = 0) = 1, \end{cases}$$

then (46) implies that

$$\mathbb{E}(\inf\{n \geq 0 : \|S_n^\alpha\| < R_{n+1}^\alpha\}) < \infty. \quad (48)$$

A similar argument shows that for some $r^\perp < \infty$ and with

$$R_n^\perp := \begin{cases} r + \nu_n & \text{if } \mathbb{P}(\zeta_1^\perp = 0) < 1, \\ r + r^\perp & \text{if } \mathbb{P}(\zeta_1^\perp = 0) = 1, \end{cases}$$

if (46) holds, then also

$$\mathbb{E}(\inf\{n \geq 0 : \|S_n^\perp\| < R_{n+1}^\perp\}) < \infty. \quad (49)$$

Using Lemma 3.1 if $\mathbb{E}\zeta_1^\alpha > 0$ or Lemma 3.4 if $\mathbb{E}\zeta_1^\alpha = 0$, for the process $(S_n^\alpha, R_n^\alpha)_{n \geq 0}$, noting that (47) implies that condition (53) is in force, we see that for (48) to hold we must have

$$x_0^\alpha < M^\alpha, \quad (50)$$

for some $M^\alpha > 0$. Similarly, since $\mathbb{E}\zeta_1^\perp = 0$ and (47) holds, we may use Lemma 3.4 for the process $(S_n^\perp, R_n^\perp)_{n \geq 0}$ to obtain $M^\perp > 0$ such that if (49) holds then

$$|x_0^\perp| < M^\perp. \quad (51)$$

Combining (50) and (51) we have $x_0 \in \mathcal{R}(-\hat{\alpha}, M)$ for $M := \max\{M^\alpha, M^\perp\}$ as desired. \square

Proof. For $m \geq 1$, let $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m$ be the recurrent irreducible state classes of the automaton $(Q_n)_{n \geq 0}$. If $q_0 \in \mathcal{S}_l$ for some $l \in \{1, \dots, m\}$, then the process $(X_n, Q_n)_{n \geq 0}$ is a single scout process with state space \mathcal{S}_l and transition function Π^l which is the proper restriction of Π to \mathcal{S}^l . Moreover, such a scout process has an irreducible automaton. Therefore, by Lemma 2.10 there exists $\hat{\alpha}_l$ and M_l such that if (9) holds then

$$x_0 \in \mathcal{R}(\hat{\alpha}_l, M_l). \quad (52)$$

For any other $q_0 \in \mathcal{S}$, there exists $l \in \{1, \dots, m\}$, $q' \in \mathcal{S}^l$, $n \leq |\mathcal{S}|$ and $x' \in \mathbb{Z}^2$ with $\|x' - x_0\| \leq n$ such that

$$\mathbb{P}_{(x_0, q_0)}(X_n = x', Q_n = q') > 0$$

If (9) holds, then by the Markov property and (52) we must have that $x' \in \mathcal{R}(\hat{\alpha}_l, M_l)$ which implies that $x_0 \in \mathcal{R}(\hat{\alpha}_l, M_l + |\mathcal{S}|)$. Taking $M := \max_{l \leq m} M_l + |\mathcal{S}|$ we obtain (10) as desired. \square

3 Random Walk Hitting Time Estimates

In this section we state and prove the random walk estimates, which are used in the proof of the main theorems.

3.1 One Random Walk with a Stretched Exponential Look-Around

Fix $\delta > 0$ and let $((\zeta_k, R_k) : k = 1, \dots)$ be a sequences of discrete i.i.d. random pairs, taking values in $\mathbb{R} \times [1, \infty)$ and satisfying

$$\mathbb{P}(|\zeta_1| + R_1 > u) \leq \frac{1}{\delta} e^{-u^\delta}, \quad u \geq 0. \quad (53)$$

Notice that we do not insist that for a given k , the random variables ζ_k, R_k are independent of each other. We shall think of ζ_k as the spatial displacement of a random walk in the k -th step and of R_k as the radius of a ball (interval) around the position of the walk at time $k - 1$, inside which the walk is allowed to “peek” at this time. We therefore fix also $s_0 \in \mathbb{R}$ and define for all $n \geq 0$,

$$S_n := s_0 + \sum_{k=1}^n \zeta_k,$$

The process $(S_n, R_n)_{n \geq 0}$ will be referred to as a *look-around* random walk.

Lemma 3.1. *Let $(S_n, R_n)_{n \geq 0}$ be the look-around random walk process defined above and suppose that (53) holds. If $\mathbb{E}(\zeta_1) > 0$, then there exists $r > 0$, such that for all $x < s_0 - r$,*

$$\mathbb{P}(\inf\{n \geq 0 : |S_n - x| \leq R_{n+1}\} = \infty) > 0. \quad (54)$$

Proof. Let $\mu := \mathbb{E}(\zeta_k) > 0$. By the Strong Law of Large numbers $\frac{S_n}{n} \rightarrow \mu$ as $n \rightarrow \infty$ almost surely. Therefore we may find m large enough such that

$$\mathbb{P}(S_n > \mu n/2 : n \geq m) > 1/2. \quad (55)$$

On the other hand, by union bound, for possibly larger m we have,

$$\mathbb{P}(\exists n \geq m : R_{n+1} > n^{1/3}) \leq \sum_{n=m}^{\infty} \frac{1}{\delta} e^{-n^{\delta/3}} \leq 1/4. \quad (56)$$

Finally, since $\min\{S_n - R_{n+1} : n = 1, \dots, m\} > -\infty$ with probability 1, it follows that we can find x_0 small enough such that for all $x \leq x_0$,

$$\mathbb{P}(|S_n - x| \leq R_{n+1} : n = 0, \dots, m) \leq 1/8. \quad (57)$$

The event in (54) is implied by the intersection of the event on the left hand side of (55) with the complements of the events on the left hand side of (56) and (57). The above shows that this has positive probability as required. \square

Next we deal with the case $\mathbb{E}(\zeta_1) = 0$.

Lemma 3.2. *Let $(S_n, R_n)_{n \geq 0}$ be the look-around random walk process defined above and suppose that (53) holds. If $\mathbb{E}(\zeta_1) = 0$, but $\mathbb{P}(\zeta_1 = 0) < 1$, then there exists $C > 0$, such that for all $u \geq 1$ and $x \in \mathbb{R}$ satisfying $0 < x - s_0 < u^{1/4}$,*

$$\mathbb{P}(\inf\{n \geq 0 : S_n + R_{n+1} \geq x\} \geq u) \leq C \frac{x - s_0}{\sqrt{u}}. \quad (58)$$

Proof. The probability on the left hand side of (58) is bounded above by

$$\mathbb{P}(\inf\{n \geq 0 : S_n \geq x\} \geq u)$$

and the result follows immediately from Theorem 5.1.7 of [27]. \square

The opposite direction is given by

Lemma 3.3. *Let $(S_n, R_n)_{n \geq 0}$ be the look-around random walk process defined above and suppose that (53) holds. Suppose also that $\mathbb{E}(\zeta_1) = 0$ and if $\mathbb{P}(\zeta_1 = 0) = 1$, then $\mathbb{P}(R_1 < M) = 1$ for some $M > 0$. Then there exist $r > 0$ and $C > 0$, such that for all $u \geq 1$ and $x \geq s_0 + r$,*

$$\mathbb{P}(\inf\{n \geq 0 : S_n + R_{n+1} \geq x\} \geq u) \geq \frac{C}{\sqrt{u}}. \quad (59)$$

Proof. For simplicity of the argument, we shall assume that ζ_1 is supported on the integers. The argument for a general (discrete, Real) distribution is similar. Without loss of generality we can assume that $s_0 = -1$ and $x = r - 1$, where $r > 0$ will be determined later. If $\mathbb{P}(\zeta_1 = 0) = 1$, then (59) will clearly hold, once $r = M + 1$. Otherwise, set

$$\eta_x := \inf\{n \geq 0 : S_n \geq x\}.$$

From Theorem 5.1.7 of [27] we have that for all $n \geq 0$ and $x \in [0, n^{1/4}]$,

$$C(x+1)/\sqrt{n} \leq \mathbb{P}(\eta_x > n) \leq C'(x+1)/\sqrt{n} \quad (60)$$

At the same time from Theorem 7 of [3], we have that for all $y \in [-n^{1/4}, -1]$,

$$\mathbb{P}(\eta_0 > n, S_n = y) \leq C|y|/n^{3/2}.$$

Then for all $k_0 \geq 1$ and $n > k_0$,

$$\begin{aligned} & \mathbb{P}(\eta_0 > n, \exists k \in [k_0, n] : S_k \geq -(\log k)^{3/\delta}) \\ & \leq \sum_{k=k_0}^n \sum_{y=-(\log k)^{3/\delta}}^{-1} \mathbb{P}(\eta_0 > k, S_k = y) \mathbb{P}(\eta_{-y} > n - k) \\ & \leq C \left(\sum_{k=k_0}^{n/2} k^{-3/2} (\log k)^{9/\delta} n^{-1/2} + \sum_{k=n/2}^{n-n^{1/2}} n^{-3/2} (\log n)^{9/\delta} (n - k)^{-1/2} + n^{1/2} n^{-3/2} (\log n)^{9/\delta} \right) \\ & \leq C n^{-1/2} k_0^{-1/3}. \end{aligned} \tag{61}$$

Now let $K := \inf \{k \geq 1 : |\zeta_m| + R_{m+1} \leq (\log(m \vee k))^{2/\delta}, \forall m \in [0, n]\}$. Then for all $k_1 \geq 2$ and $n > k_1$,

$$\begin{aligned} \mathbb{P}(\eta_0 > n, K \in [k_1, n/2]) &= \sum_{k=k_1}^{n/2} \mathbb{P}(\eta_0 > n, K = k) \\ &\leq \sum_{k=k_1}^{n/2} \mathbb{P}\left((\log(k-1))^{2/\delta} < \min_{m \leq k-1} (|\zeta_m| + R_{m+1}) \leq \max_{m \leq k} (|\zeta_m| + R_{m+1}) \leq (\log k)^{2/\delta}, \eta_0 > n\right) \\ &\leq C \sum_{k=k_1}^{n/2} k e^{-C'(\log(k-1))^2} \frac{k(\log k)^{2/\delta} + 1}{\sqrt{n-k}} \leq C n^{-1/2} e^{-C''(\log(k_1))^2}. \end{aligned} \tag{62}$$

Above to derive the one before last inequality, we have conditioned on \mathcal{F}_k and used the Markov property together with (53), (60), union bound and the fact that

$$\left\{ \max_{m \leq k} (|\zeta_m| + R_{m+1}) \leq (\log k)^{2/\delta} \right\} \subseteq \left\{ S_k \geq -k(\log k)^{2/\delta} \right\}.$$

Also by union bound and (53),

$$\begin{aligned} \mathbb{P}(K > n/2) &\leq \mathbb{P}(|\zeta_m| + R_{m+1} > (\log(n/2))^{2/\delta} : m \in [0, n]) \\ &\leq C n e^{-C'(\log n)^2} \leq e^{-C''(\log n)^2}, \end{aligned} \tag{63}$$

Combining (62) and (63) we get

$$\mathbb{P}(\eta_0 > n, K \geq k_1) \leq n^{-1/2} e^{-C'(\log(k_1))^2} + e^{-C''(\log n)^2}. \tag{64}$$

Subtracting the probabilities on the left hand sides of (61), (64) with $k_0 = k_1$ from the probability in (60) for $x = 0$, and using the derived upper bounds for the former and the lower bound

for the latter, we have

$$\begin{aligned} & \mathbb{P}(S_m < -(\log m)^{3/\delta} : m \in [k_0, n], \quad |\zeta_m| + R_{m+1} < (\log(m \vee k_0))^{2/\delta} : m \in [0, n]) \\ & \geq n^{-1/2} (C - C' k_0^{-1/3} - e^{-C''(\log k_0)^2} - n^{1/2} e^{-C'''(\log n)^2}) \end{aligned}$$

We may now find k_0 large enough such that for all n large enough, the right hand side above will be at least $Cn^{-1/2}$ for some $C > 0$. But the event on the left hand side implies that

$$\{S_m + R_{m+1} \leq x : m \in [0, n]\}$$

for all $x > k_0(\log k_0)^{2/\delta}$. Setting $r = k_0(\log k_0)^{2/\delta}$, this shows (59) for all $x \geq r$ as desired. \square

The following is an immediate consequence of Lemma 3.3.

Lemma 3.4. *Let $(S_n, R_n)_{n \geq 0}$ be the look-around random walk process defined above and suppose that (53) holds, $\mathbb{E}(\zeta_1) = 0$ and if $\mathbb{P}(\zeta_1 = 0) = 1$, then $\mathbb{P}(R_1 < M) = 1$ for some $M > 0$. Then there exist $r > 0$, such that if $|s_0 - x| > r$, then*

$$\mathbb{E}(\inf\{n \geq 0 : |S_n - x| \leq R_{n+1}\}) = \infty. \quad (65)$$

Proof. By reversing the steps, it is enough to show the result for $x \geq s_0 + r$. But then, using the tail formula for expectation, we sum (59) from $u = 1$ to ∞ and conclude (65) for all such x if r is large enough. \square

Lemma 3.5. *Let $(S_n, R_n)_{n \geq 0}$ be the look-around random walk process defined above. Suppose that (53) holds. Then if*

$$|\{x \in \mathbb{R} : \mathbb{E}(\inf\{n \geq 0 : |S_n - x| \leq R_{n+1}\}) = \infty\}| < \infty, \quad (66)$$

we must have

$$\mathbb{P}(\zeta_1 = 0) = 1.$$

Proof. If $\mathbb{E}\zeta_1 \neq 0$, then by Lemma 3.1 we have that (66) must be false. At the same time, if $\mathbb{E}\zeta_1 = 0$ and $\mathbb{P}(\zeta_1 = 0) < 1$, then by Lemma 3.4 the inequality in (66) must be again false. It follows that $\mathbb{P}(\zeta_1 = 0) = 1$ as required. \square

In the next two lemmas, the “look-around” feature of the random walk is not used. The first one includes standard hitting time estimates for random walks. We omit the proof, as it is standard. For $\rho > 0$, let

$$\tau_\rho := \begin{cases} \inf\{n \geq 0 : |S_n| > \rho\}, & \text{if } \mathbb{E}\zeta_1 = 0, \\ \inf\{n \geq 0 : S_n > \rho\}, & \text{if } \mathbb{E}\zeta_1 > 0, \\ \inf\{n \geq 0 : S_n < -\rho\}, & \text{if } \mathbb{E}\zeta_1 < 0. \end{cases} \quad (67)$$

Then,

Lemma 3.6. *Let $(S_n, R_n)_{n \geq 0}$ be the look-around random walk defined above and suppose that $\mathbb{P}(\zeta_1 = 0) < 1$. Then for all $\rho > 0$, there exists $\delta > 0$, such that for all $s_0 \in \mathbb{R}$,*

$$\mathbb{P}(\tau_\rho > u) \leq \delta^{-1} e^{-\delta(u - |s_0|)}.$$

The next lemma is the result of a standard application of the exponential Chebychev inequality.

Lemma 3.7. *Let $(S_n, R_n)_{n \geq 0}$ be the look-around random walk described above and suppose that $\mathbb{E}\zeta_1 = s_0 = 0$. Then for all $\mu > 0$, there exists $\delta > 0$, such that if $n \geq 0$ and $y \geq \mu n$ then,*

$$\mathbb{P}(S_n \geq y) \leq e^{-\delta y}.$$

Proof. Let $L(t) := \log \mathbb{E} e^{t\zeta_1}$ be the log moment generating of ζ_1 , whose existence and (Real) analyticity in a neighborhood of 0 is guaranteed by condition (53). Since $L(0) = 0$ and $L'(0) = \mathbb{E}\zeta_1 = 0$. It follow by Taylor expansion of L around 0, that for any $\mu > 0$, we may find $t_0 > 0$ and $\delta_0 > 0$, such that

$$L(t_0) - \mu t_0 < -\delta_0.$$

Then for n and y as in the conditions of the lemma, by the exponential Chebychev inequality,

$$\begin{aligned} \mathbb{P}(S_n \geq y) &\leq \exp\{nL(t_0) - n\mu t_0 - (y - n\mu)t_0\} \\ &\leq \exp\{-\delta_0 n - (y - n\mu)t_0\} \leq \exp\{-\delta y\}, \end{aligned}$$

where $\delta = \min\{t_0, \delta_0/\mu\}$. □

3.2 Two Time-Varying Random Walks

In this sub-section we consider two random walks of the type considered in sub-section 3.1, only that in addition, each walk also has a “time” component. The latter can be used to “synchronize” between the two walks.

As in the previous subsection, we fix $\delta > 0$ and for $i = 1, 2$, let $((\zeta_k^i, \nu_k^i, R_k^i) : k = 1, \dots)$ be two independent sequences of i.i.d. discrete random triplets, taking values in $\mathbb{R} \times \mathbb{Z}_{\geq 1} \times [1, \infty)$ and satisfying the following conditions:

1. $\mathbb{P}(|\zeta_1^i| + \nu_1^i + R_1^i > u) \leq \frac{1}{\delta} e^{-u^\delta}$.
2. If ζ_1^i/ν_1^i is non-random, then $\mathbb{P}(\nu_1^i = 1) = 1$ and $\mathbb{P}(|R_1^i| < 1/\delta) = 1$.

Again, we do not insist that for a given i and k , the random variables $\zeta_k^i, \nu_k^i, R_k^i$ are independent of each other. We shall think of ν_k^i as the time it took walk i to make step k and of R_k^i as its “look-around” radius (in the sense of the previous subsection). For $i = 1, 2$, we now fix also $s_0^i \in \mathbb{R}$ and define the random walk $((S_n^i, T_n^i) : n = 0, 1, \dots)$ by

$$S_n^i := s_0^i + \sum_{k=1}^n \zeta_k^i, \quad T_n^i := \sum_{k=1}^n \nu_k^i; \quad n = 0, 1, \dots$$

Setting also $R_0^i = 0$, the resulting process $((S_n^i, T_n^i, R_n^i) : n = 0, 1, \dots)$ will be referred to as a *time-varying look-around random walk*. Its *effective drift* is then given by

$$d^i := (\mathbb{E}\zeta_1^i)/(\mathbb{E}\nu_1^i). \quad (68)$$

We now define various hitting times. First, to translate back “time” to number of steps, we set for $i = 1, 2$ and $m \geq 0$ the random variable,

$$K_m^i := \sup \{k \geq 0 : T_k^i \leq m\}.$$

Now, for a subset $A \subseteq \mathbb{R}$, the hitting time of A by S_n^i is

$$\tau_A^i := \inf \{m \geq 0 : d(S_{K_m^i}^i, A) \leq R_{K_m^i+1}^i\}, \quad (69)$$

where for $x \in \mathbb{R}$ we use the usual definition $d(x, A) := \inf\{\|x - y\| : y \in A\}$. The first time the look-around balls (intervals) of the walks intersect, is defined via

$$\sigma := \inf \{m \geq 0 : |S_{K_m^1}^1 - S_{K_m^2}^2| \leq R_{K_m^1+1}^1 + R_{K_m^2+1}^2\}.$$

Finally, we also set $s_0^\Delta := s_0^1 - s_0^2$ and $d^\Delta := d^1 - d^2$.

The following proposition is the main product of this subsection.

Proposition 3.8. *For $i = 1, 2$, let the time-varying look-around random walks $((S_n^i, T_n^i, R_n^i) : n = 0, 1, \dots)$ be as defined above. Then, we may find $\rho > 0$, such that whenever:*

$$\begin{aligned} |s_0^\Delta| &> \rho && \text{if } d^\Delta = 0, \\ s_0^\Delta &> \rho && \text{if } d^\Delta > 0, \\ s_0^\Delta &< -\rho && \text{if } d^\Delta < 0, \end{aligned} \quad (70)$$

there exist $x, y \in \mathbb{Z}$ satisfying $-\infty < x < y - 2 < \infty$, such that

$$\mathbb{E} \min(\sigma, \tau_{[x,y]}^1, \tau_{[x,y]}^2) = \infty. \quad (71)$$

Proof. To simplify the arguments below, we shall assume that ζ_1^i, R_1^i are supported on \mathbb{Z} . The argument for general (discrete, Real) distributions is analogous. The proof follows by case-analysis of d^1 and d^2 . By switching between walk 1 and 2 or negating the two walks simultaneously, we do not lose any generality by considering only the cases:

- $d^1 \geq 0, d^2 \leq 0$,
- $d^1 > d^2 > 0$,
- $d^1 = d^2 > 0$.

If $d^1 \geq 0$ and $d^2 \leq 0$, then $d^\Delta \geq 0$ and $\mathbb{E}\zeta_1^1 \geq 0$, $\mathbb{E}\zeta_1^2 \leq 0$. We define now for $A \subset \mathbb{R}$, $i = 1, 2, \Delta$ the hitting time of A by walk i as

$$N_A^i := \inf \{n \geq 0 : d(S_n^i, A) \leq R_{n+1}^i\}.$$

This is analog to τ_A^i only that time is measured in steps.

Then, by Lemma 3.1 and Lemma 3.3, we may find $\rho > 0$ large enough such that if $s_0^\Delta = s_0^1 - s_0^2 > \rho$, then there exist $x, y \in \mathbb{Z}$ such that $s_0^1 < x < y - 2 < s_0^2$ and for all $u \geq 0$,

$$\mathbb{P}(N_{[x,y]}^1 > u) \geq \frac{C}{\sqrt{u}}, \quad \mathbb{P}(N_{[x,y]}^2 > u) \geq \frac{C'}{\sqrt{u}},$$

Since $\tau_{[x,y]}^i \geq N_{[x,y]}^i$, it follows that for all $u \geq 0$, also

$$\mathbb{P}(\tau_{[x,y]}^1 > u) \geq \frac{C}{\sqrt{u}}, \quad \mathbb{P}(\tau_{[x,y]}^2 > u) \geq \frac{C'}{\sqrt{u}}.$$

Now, it is not difficult to see that for such s_0^Δ ,

$$\sigma \geq \min\{\tau_{[x,y]}^1, \tau_{[x,y]}^2\}.$$

It then follows that

$$\mathbb{E} \min\{\sigma, \tau_{[x,y]}^1, \tau_{[x,y]}^2\} = \mathbb{E} \min\{\tau_{[x,y]}^1, \tau_{[x,y]}^2\}.$$

Since the two walks are independent, by the tail formula for expectation,

$$\mathbb{E} \min\{\sigma, \tau_{[x,y]}^1, \tau_{[x,y]}^2\} = \sum_{u=1}^{\infty} \mathbb{P}(\tau_{[x,y]}^1 \geq u) \mathbb{P}(\tau_{[x,y]}^2 \geq u) \geq \sum_{u=1}^{\infty} \frac{CC'}{u} = \infty.$$

Moving to the case $d^1 > d^2 > 0$. Here $d^\Delta > 0$. By the law of large numbers, for all $\epsilon > 0$, we may find n_0 large enough, such that with probability at least $1 - \epsilon$ for all $n \geq n_0$, $i = 1, 2$, the following holds:

$$\begin{aligned} n(\mathbb{E}\zeta^i - \epsilon) &\leq S_n^i \leq n(\mathbb{E}\zeta^i + \epsilon), \\ n(\mathbb{E}\nu^i - \epsilon) &\leq T_n^i \leq n(\mathbb{E}\nu^i + \epsilon). \end{aligned}$$

At the same time, by increasing n_0 if needed, we have

$$\mathbb{P}(\exists n \geq n_0 : R_n^i > n^{1/3}) \leq \sum_{n=n_0}^{\infty} \delta^{-1} e^{-\delta n^{1/3}} \leq e^{-Cn_0^{1/3}} \leq \epsilon.$$

Combining the last three displays, for all $\epsilon > 0$, we may find m_0 , such that with probability at least $1 - \epsilon$, for all $m \geq m_0$,

$$m(d^i - \epsilon) \leq S_{K_m^i}^i \leq m(d^i + \epsilon), \quad R_{K_m^i}^i \leq \epsilon m. \quad (72)$$

By finiteness almost surely of the random variables $K_{m_0}^i, |\zeta_n^i| + R_n^i$, for all $i = 1, 2$ and $n \geq 1$, for any m_0 and $\epsilon > 0$, we may find $r > 0$ large enough, such that with probability at least $1 - \epsilon$,

$$\sum_{n=1}^{K_{m_0}^i+1} |\zeta_n^i| + R_n^i < r. \quad (73)$$

Combining the above, we first choose $0 < \epsilon < \min\{d_1 - d_2, d_2, 1\}/5$, then find m_0 large enough and finally $r > 0$ such that both (72) and (73) hold with probability at least $1 - \epsilon$. It follows that if $s_0^\Delta > 3r$ and $x, y \in \mathbb{Z}$ satisfy $x + 2 < y < s_0^2 - 2r$, then with probability at least $1 - 2\epsilon$

$$\tau_{[x,y]}^1 = \infty, \quad \tau_{[x,y]}^2 = \infty, \quad \sigma = \infty,$$

which, of course, implies (71) .

Turning to the hardest case $d_1 = d_2 =: d > 0$. For $\rho > 2$ to be determined later, let us assume that $s_0^\Delta > \rho$ and without loss of generality also that $s_0^2 = 0$. For $i = 1, 2$ and $n \geq 0$, set

$$Y_n^i := S_n^i - dT_n^i = s_0^i + \sum_{k=1}^n (\zeta_k^i - d\nu_k^i).$$

Let also $z := s_0^1/2$ and define

$$\sigma^1 := \inf\{n \geq 0 : Y_n^1 - d\nu_{n+1}^1 - R_{n+1}^1 \leq z\}, \quad \sigma^2 := \inf\{n \geq 0 : Y_n^2 + R_{n+1}^2 \geq z\}.$$

Observe that

$$\begin{aligned} \{\sigma^1 > n\} &\subseteq \{S_{K_m^1}^1 - R_{K_m^1+1}^1 > dm + z : m = 0, \dots, T_n^1\}, \\ \{\sigma^2 > n\} &\subseteq \{S_{K_m^2}^2 + R_{K_m^2+1}^2 < dm + z : m = 0, \dots, T_n^2\}. \end{aligned}$$

Consequently,

$$\{\sigma^1 > n, \sigma^2 > n\} \subseteq \{\sigma > T_n^1 \wedge T_n^2\} \subseteq \{\sigma > n\}.$$

Similarly if $x, y \in \mathbb{Z}$ such that $x + 2 < y < 0$, then

$$\{\sigma^1 > n, \sigma^2 > n, N_{[x,y]}^2 > n\} \subseteq \{\sigma > n, \tau_{[x,y]}^1 > n, \tau_{[x,y]}^2 > n\} = \{\min\{\sigma, \tau_{[x,y]}^1, \tau_{[x,y]}^2\} > n\}.$$

Thanks to the independence between the two walks and the tail formula for expectation, to show (71), it is therefore enough to prove

$$\sum_{n=0}^{\infty} \mathbb{P}(\sigma^1 > n) \mathbb{P}(\sigma^2 > n, N_{[x,y]}^2 > n) = \infty. \quad (74)$$

To this end, first observe that (68) implies that $\mathbb{E}\zeta_1^i - d\nu_1^i = 0$. This makes Y_n^i a random walk with 0 drift starting from s_0^i . Moreover, by our assumptions on $(\zeta_1^i, \nu_1^i, R_1^i)$, the processes $(Y_n^1, R_n^1 + d\nu_n^1)_{n \geq 0}$ and $(Y_n^2, R_n^2)_{n \geq 0}$ are look-around random walks, which satisfy the conditions in Lemma 3.3. Therefore, by the lemma, if $\rho > 0$ is chosen large enough, we have

$$\mathbb{P}(\sigma^i > u) \geq C/\sqrt{u}, \quad u \geq 0, \quad i = 1, 2. \quad (75)$$

This handles the first probability in (74). For the second, we write

$$\mathbb{P}(\sigma^2 > n, N_{[x,y]}^2 > n) = \mathbb{P}(\sigma^2 > n) - \mathbb{P}(\sigma^2 > n, N_{[x,y]}^2 \leq n) \quad (76)$$

By (75), the first term is lower bounded by C/\sqrt{n} once ρ is chosen large enough. We wish to show now that the second term can be upper bounded by $C/(2\sqrt{n})$ by choosing then y small enough.

Thanks to Lemma 3.7, writing μ for $\mathbb{E}\zeta_1^2$, we have for all $k \geq 0$ and $y < 0$,

$$\begin{aligned} \mathbb{P}(S_k^2 - R_{k+1}^2 \leq y) &\leq \mathbb{P}(S_k^2 \leq k\mu/2 + y/2) + \mathbb{P}(R_{k+1}^2 \geq k\mu/2 - y/2) \\ &\leq Ce^{-C'(k-y)}. \end{aligned} \quad (77)$$

Therefore,

$$\mathbb{P}(N_{[x,y]}^2 > n^{1/16}) \leq \sum_{k=n^{1/16}}^{\infty} \mathbb{P}(S_k^2 - R_{k+1}^2 \leq y) \leq e^{-Cn^{1/16}}.$$

At the same time,

$$\begin{aligned} \mathbb{P}(\exists k \leq n^{1/16} : |S_k^2| + |T_k^2| + R_{k+1}^2 > n^{1/4}) &\leq \mathbb{P}(\exists k \leq n^{1/16} + 1 : |\zeta_k^2| + \nu_k^2 + R_k^2 > n^{1/16}) \\ &\leq (n^{1/16} + 1)e^{-Cn^{1/16}} \leq e^{-C'n^{1/16}}. \end{aligned}$$

It follows that the second term on the right hand side of (76) is bounded above by

$$e^{-Cn^{-1/16}} + \sum_{k=1}^{n^{1/16}} \sum_{w=-2n^{1/4}}^y \sum_{m=k}^{n^{1/4}} \mathbb{P}(S_k^2 - R_{k+1}^2 = w, T_k^2 = m) \mathbb{P}(\sigma^2 > n - k \mid Y_0^2 = w - dm), \quad (78)$$

where conditioning in the second probability is only formal and means that $(Y_n^2)_{n \geq 0}$ is redefined so that $Y_0^2 = w - dm$.

In the range of the sums, we can use Lemma 3.2 to upper bound the second probability by

$$C \frac{z - w + dm}{\sqrt{n - k}} \leq C' \frac{z - w + dm}{\sqrt{n}}.$$

Plugging this into (78) and removing some of the restrictions on the sums, we get as an upper bound,

$$e^{-Cn^{-1/16}} + C'n^{-1/2} \sum_{k=1}^{\infty} \mathbb{E} \left((z - S_k^2 + R_{k+1}^2 + dT_k^2) 1_{\{S_k^2 - R_{k+1}^2 \leq y\}} \right). \quad (79)$$

Using Cauchy-Schwartz and (77), the last expectation can be further bounded above by

$$C(z^2 + \mathbb{E}|S_k^2|^2 + \mathbb{E}|R_{k+1}^2|^2 + \mathbb{E}|T_k^2|^2)^{1/2} (\mathbb{P}(S_k^2 - R_{k+1}^2 \leq y))^{1/2} \leq C(z + k)e^{-C'(k-y)}.$$

Consequently, (79) is bounded above by

$$e^{-Cn^{-1/16}} + C'n^{-1/2}(z + 1)e^{C''y} \sum_{k=1}^{\infty} ke^{-C'''k} \leq e^{-Cn^{-1/16}} + C'n^{-1/2}(z + 1)e^{C''y}.$$

Choosing first ρ (and hence z) large enough and then y small enough, by (75) for $i = 2$ and (76), we can indeed ensure that

$$\mathbb{P}(\sigma^2 > n, N_{[x,y]}^2 > n) \geq C/\sqrt{n}.$$

Together with (75) for $i = 1$ we get (74) and hence also (71). \square

4 One Scout on \mathbb{Z}^1

The arguments leading to the proof of Theorem 2.1 can be adapted to yield a proof for the case $d = 1$.

Theorem 4.1. *Let $(X_n, Q_n)_{n \geq 0}$ be a single scout process on \mathbb{Z} with protocol $\mathcal{P} = \langle 1, \mathcal{S}, 0, q_0, \Pi \rangle$. Then there exists some grid point $x \in \mathbb{Z}$ of which the expected hitting time is infinite, namely,*

$$\mathbb{E}(\inf\{n \geq 0 : X_n = x\}) = \infty. \quad (80)$$

Proof. Recalling that the automaton process $(Q_n)_{n \geq 0}$ is a Markov chain on \mathcal{S} , we let $\mathcal{S}_1, \dots, \mathcal{S}_l$ for $l \geq 1$, be its irreducible recurrent state classes. Suppose first that $q_0 \in \mathcal{S}_m$ for some $m \in \{1, \dots, l\}$, let $T_0 = 0$ and for $k = 1, \dots$, set

$$\begin{aligned} T_k &:= \inf\{n > T_{k-1} : Q_n = q_0\}, \\ \zeta_k &:= X_{T_k} - X_{T_{k-1}}, \\ \nu_k &:= T_k - T_{k-1}. \end{aligned} \quad (81)$$

By the Markov property and spatial homogeneity of the process $(X_n, Q_n)_{n \geq 0}$, the pairs $(\zeta_k, \nu_k)_{k \geq 1}$ are i.i.d. Moreover, standard Markov chain theory shows that

$$\mathbb{P}(|\zeta_k| + \nu_k > u) \leq C^{-1}e^{-Cu}, \quad (82)$$

for some $C > 0$. In particular, if we set for $n \geq 0$,

$$S_n := x_0 + \sum_{k=1}^n \zeta_k = X_{T_n}, \quad (83)$$

then $(S_n)_{n \geq 0}$ is a random walk on \mathbb{Z} . If $\mathbb{P}(\zeta_1 = 0) = 1$, then we appeal to Lemma 2.6 to obtain $r > 0$, such that $\mathbb{P}(|X_{T_1}| < r) = 1$ and accordingly we set $R_n := r$ for all $n \geq 1$. Otherwise, we set $R_n := \nu_n$. In both cases we then have

$$\inf\{n \geq 0 : X_n = x\} \geq \inf\{n \geq 0 : |S_n - x| \leq R_{n+1}\}. \quad (84)$$

Now, in light of (82), the process $(S_n, R_n)_{n \geq 0}$ is a look-around random walk of the type treated in Subsection 3.1, starting from 0. Moreover, by definition if $\mathbb{P}(\zeta_1 = 0) = 1$, then R_n is bounded by a deterministic quantity. By employing Lemma 3.1 or Lemma 3.4, we conclude that there exist infinitely many $x \in \mathbb{Z}$ such that

$$\mathbb{E}(\inf\{n \geq 0 : X_n = x\}) \geq \mathbb{E}(\inf\{n \geq 0 : |S_n - x| \leq R_{n+1}\}) = \infty. \quad (85)$$

In particular, this shows (80).

If q_0 does not belong to any of \mathcal{S}_m for $m \in \{1, \dots, l\}$, then we set

$$\sigma := \inf\{n : Q_n \in \cup_{m=1}^l \mathcal{S}_m\} \quad (86)$$

and let L be such that $Q_\sigma \in \mathcal{S}_L$. Standard Markov chain theory gives that $\sigma < \infty$ almost surely and hence L is well defined. Since σ is a stopping time, conditional on \mathcal{F}_σ , the process $(X_{\sigma+n}, Q_{\sigma+n})_{n \geq 0}$ is distributed as the original process starting from $x_0 = X_\sigma$ and $q_0 = Q_\sigma$. Moreover, since σ is finite, the number of visited grid points $x \in \mathbb{Z}$ up to this time is finite. Repeating the argument above with $m = L$, there is at least one (random, \mathcal{F}_σ -measurable) $x' \in \mathbb{Z}$ which was not visited up to time σ and such that

$$\mathbb{E}(\inf\{n \geq 0 : X_{\sigma+n} = x'\} \mid \mathcal{F}_\sigma) = \infty. \quad (87)$$

This readily implies (80) for some (deterministic) $x \in \mathbb{Z}$. \square

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